Smooth Livšic regularity for piecewise expanding maps

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Abstract

We consider the regularity of measurable solutions χ to the cohomological equation

$$\phi = \chi \circ T - \chi,$$

where (T, X, μ) is a dynamical system and $\phi\colon X \to \mathbb{R}$ is a C^k valued cocycle in the setting in which $T\colon X \to X$ is a piecewise C^k Gibbs–Markov map, an affine β-transformation of the unit interval or more generally a piecewise C^k uniformly expanding map of an interval. We show that under mild assumptions, bounded solutions χ possess C^k versions. In particular we show that if (T, X, μ) is a β-transformation then χ has a C^k version, thus improving a result of Pollicott et al. [23].

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1 Introduction

In this note we consider the regularity of solutions χ to the cohomological equation

$$\phi = \chi \circ T - \chi \tag{1}$$

where (T, X, μ) is a dynamical system and $\phi\colon X \to \mathbb{R}$ is a C^k valued cocycle. In particular we are interested in the setting in which $T\colon X \to X$ is a piecewise C^k Gibbs–Markov map, an affine β-transformation of the unit interval

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or more generally a piecewise C^k uniformly expanding map of an interval. Rigidity in this context means that a solution χ with a certain degree of regularity is forced by the dynamics to have a higher degree of regularity. Cohomological equations arise frequently in ergodic theory and dynamics and, for example, determine whether observations ϕ have positive variance in the central limit theorem and and have implication for other distributional limits (for examples see [20, 2]). Related cohomological equations to Equation (1) decide on stable ergodicity and weak-mixing of compact group extensions of hyperbolic systems [11, 20, 19] and also play a role in determining whether two dynamical systems are (Hölder, smoothly) conjugate to each other.

Livšic [13, 14] gave seminal results on the regularity of measurable solutions to cohomological equations for Abelian group extensions of Anosov systems with an absolutely continuous invariant measure. Theorems which establish that a priori measurable solutions to cohomological equations must have a higher degree of regularity are often called measurable Livšic theorems in honor of his work.

We say that $\chi\colon X \to \mathbb{R}$ has a C^k version (with respect to μ) if there exists a C^k function $h\colon X \to \mathbb{R}$ such that $h(x) = \chi(x)$ for μ a.e. $x \in X$.

Pollicott and Yuri [23] prove Livšic theorems for Hölder $\mathbb{R}$-extensions of β-transformations ($T\colon [0,1) \to [0,1)$, $T(x) = \beta x \pmod 1$ where $\beta > 1$) via transfer operator techniques. They show that any essentially bounded measurable solution χ to Equation (1) is of bounded variation on $[0, 1-\epsilon)$ for any $\epsilon > 0$. In this paper we improve this result to show that measurable coboundaries χ for C^k $\mathbb{R}$-valued cocycles ϕ over β-transformations have C^k versions (see Theorem 2).

Jenkinson [10] proves that integrable measurable coboundaries χ for $\mathbb{R}$-valued smooth cocycles ϕ (i.e. again solutions to $\phi = \chi \circ T - \chi$) over smooth expanding Markov maps T of S^1 have versions which are smooth on each partition element.

Nicol and Scott [15] have obtained measurable Livšic theorems for certain discontinuous hyperbolic systems, including β-transformations, Markov maps, mixing Lasota–Yorke maps, a simple class of toral-linked twist map and Sinai dispersing billiards. They show that a measurable solution χ to Equation (1) has a Lipschitz version for β-transformations and a simple class of toral-linked twist map. For mixing Lasota–Yorke maps and Sinai dispersing billiards they show that such a χ is Lipschitz on an open set. There is an error in [15, Theorem 1] in the setting of C^2 Markov maps — they only prove measurable solutions χ to Equation (1) are Lipschitz on each element $T\alpha$, $\alpha \in \mathcal{P}$, where $\mathcal{P}$ is the defining partition for the Markov map, and not that the solutions are Lipschitz on X, as Theorem 1 erroneously states. The

error arose in the following way: if χ is Lipschitz on $\alpha \in \mathcal{P}$ it is possible to extend χ as a Lipschitz function to $T\alpha$ by defining $\chi(Tx) = \phi(x) + \chi(x)$, however extending χ as a Lipschitz function from α to $T^2\alpha$ via the relation $\chi(T^2x) = \phi(Tx) + \chi(Tx)$ may not be possible, as $\phi \circ T$ may have discontinuities on $T\alpha$. In this paper we give an example, (see Section 3), which shows that for Markov maps this result cannot be improved on.

Gouëzel [7] has obtained similar results to Nicol and Scott [15] for cocycles into Abelian groups over one-dimensional Gibbs–Markov systems. In the setting of Gibbs–Markov system with countable partition he proves any measurable solution χ to Equation (1) is Lipschitz on each element $T\alpha$, $\alpha \in \mathcal{P}$, where $\mathcal{P}$ is the defining partition for the Gibbs–Markov map.

In related work, Aaronson and Denker [1, Corollary 2.3] have shown that if $(T, X, \mu, \mathcal{P})$ is a mixing Gibbs–Markov map with countable Markov partition $\mathcal{P}$ preserving a probability measure μ and $\phi\colon X \to \mathbb{R}^d$ is Lipschitz (with respect to a metric ρ on X derived from the symbolic dynamics) then any measurable solution $\chi\colon X \to \mathbb{R}^d$ to $\phi = \chi \circ T - \chi$ has a version $\tilde{\chi}$ which is Lipschitz continuous, i.e. there exists $C > 0$ such that $d(\tilde{\chi}(x), \tilde{\chi}(y)) \leq C\rho(x, y)$ for all $x, y \in T(\alpha)$ and each $\alpha \in \mathcal{P}$.

Bruin et al. [4] prove measurable Livšic theorems for dynamical systems modelled by Young towers and Hofbauer towers. Their regularity results apply to solutions of cohomological equations posed on Hénon-like mappings and a wide variety of non-uniformly hyperbolic systems. We note that Corollary 1 of [4, Theorem 1] is not correct — the solution is Hölder only on M_k and TM_k rather than T^jM_k for $j > 1$ as stated for reasons similar to those given above for the result in Nicol et al. [15].

2 Main results

We first describe one-dimensional Gibbs–Markov maps. Let $I \subset \mathbb{R}$ be a bounded interval, and $\mathcal{P}$ a countable partition of I into intervals. We let m denote Lebesgue measure. Let $T\colon I \to I$ be a piecewise C^k, $k \geq 2$, expanding map such that T is C^k on the interior of each element of $\mathcal{P}$ with $|T'| > \lambda > 1$, and for each $\alpha \in \mathcal{P}$, $T\alpha$ is a union of elements in $\mathcal{P}$. Let $P_n := \bigvee_{j=0}^{n} T^{-j}\mathcal{P}$ and $J_T := \frac{d(m \circ T)}{dm}$. We assume:

(i) (Big images property) There exists $C_1 > 0$ such that $m(T\alpha) > C_1$ for all $\alpha \in \mathcal{P}$.

(ii) There exists $0 < \gamma_1 < 1$ such that $m(\beta) < \gamma_1^n$ for all $\beta \in P_n$.

(iii) (Bounded distortion) There exists $0 < \gamma_2 < 1$ and $C_2 > 0$ such that $|1 - \frac{J_T(x)}{J_T(y)}| < C_2 \gamma_2^n$ for all $x, y \in \beta$ if $\beta \in P_n$.

Under these assumptions T has an invariant absolutely continuous probability measure μ and the density of μ, $h = \frac{d\mu}{dm}$ is bounded above and below by a constant $0 < C^{-1} \leq h(x) \leq C$ for m a.e. $x \in I$.

Note that a Markov map satisfies (i), (ii) and (iii) for finite partition $\mathcal{P}$.

It is proved in [15] for the Markov case (finite $\mathcal{P}$), and in [7] for the Gibbs–Markov case (countable $\mathcal{P}$) that if $\phi \colon I \to \mathbb{R}$ is Hölder continuous or Lipschitz continuous, and $\phi = \chi \circ T - \chi$ for some measurable function $\chi \colon I \to \mathbb{R}$, then there exists a function $\chi_0 \colon I \to \mathbb{R}$ that is Hölder or Lipschitz on each of the elements of $\mathcal{P}$ respectively, and $\chi_0 = \chi$ holds μ (or m) a.e. A related result to [7] is given in [4, Theorem 7] where T is the base map of a Young Tower, which has a Gibbs–Markov structure.

Fried [6] has shown that the transfer operator of a graph directed Markov system with $C^{k,\alpha}$-contractions, acting on a space of $C^{k,\alpha}$-functions, has a spectral gap. If we apply his result to our setting, letting the contractions be the inverse branches of a Gibbs–Markov map we can conclude that the transfer operator of a Gibbs–Markov map acting on C^k-functions has a spectral gap. As in Jenkinson's paper [10] and with the same proof, this gives us immediately the following proposition, which is implied by the results of Fried and Jenkinson:

Proposition 1. Let $T \colon T \to I$ be a mixing Gibbs–Markov map such that T is C^k on each partition element and $T^{-1} \colon T(\alpha) \to \alpha$ is C^k on each partition element $\alpha \in \mathcal{P}$. Let $\phi \colon I \to \mathbb{R}$ be uniformly C^k on each of the partition elements $\alpha \in \mathcal{P}$. Suppose $\chi \colon I \to \mathbb{R}$ is a measurable function such that $\phi = \chi \circ T - \chi$. Then there exists a function $\chi_0 \colon I \to \mathbb{R}$ such that χ_0 is uniformly C^k on $T\alpha$ for each partition element of $\alpha \in \mathcal{P}$, and $\chi_0 = \chi$ almost everywhere.

3 A counterexample

We remark that in general, if $\phi = \chi \circ T - \chi$, one cannot expect χ to be continuous on I if ϕ is C^k on I. We give an example of a Markov map T with Markov partition $\mathcal{P}$, a function ϕ that is C^k on I, and a function χ that is C^k on each element α of $\mathcal{P}$ such that $\phi = \chi \circ T - \chi$, yet χ has no version that is continuous on I.

Let $0 < c < \frac{1}{4}$. Put $d = 2 - 4c$. Define $T\colon [0,1] \to [0,1]$ by

$$T(x) = \begin{cases} 2x + \frac{1}{2} & \text{if } 0 \le x \le \frac{1}{4} \\ d(x - \frac{1}{2}) + \frac{1}{2} & \text{if } \frac{1}{4} < x < \frac{3}{4} \\ 2x - \frac{3}{2} & \text{if } \frac{3}{4} \le x \le 1 \end{cases}.$$

If $c = \frac{1}{8}$, then the partition

$$\mathcal{P} = \Big\{ \Big[0, \frac{1}{8}\Big], \Big[\frac{1}{8}, \frac{1}{4}\Big], \Big[\frac{1}{4}, \frac{1}{2} - \frac{1}{4d}\Big], \Big[\frac{1}{2} - \frac{1}{4d}, \frac{1}{2}\Big], \Big[\frac{1}{2}, \frac{1}{2} + \frac{1}{4d}\Big], \Big[\frac{1}{2} + \frac{1}{4d}, \frac{3}{4}\Big], \Big[\frac{3}{4}, \frac{7}{8}\Big], \Big[\frac{7}{8}, 1\Big] \Big\}$$

is a Markov partition for T. Define χ such that χ is 0 on $[\frac{1}{2} - \frac{1}{4d}, \frac{1}{2}]$ and 1 on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{4d}]$. On $[0, \frac{1}{4})$ we define χ so that $\chi(0) = 1$ and $\lim_{x \to \frac{1}{4}} \chi(x) = 0$, and on $(\frac{3}{4}, 1]$ we define χ so that $\chi(1) = 0$ and $\lim_{x \to \frac{3}{4}} \chi(x) = 1$. For any natural number k, this can be done so that χ is C^k except at the point $\frac{1}{2}$ where it has a jump. One easily check that ϕ defined by $\phi = \chi \circ T - \chi$ is C^k. This is illustrated in Figures 1–4.

4 Livšic theorems for piecewise expanding maps of an interval

Let $I = [0, 1)$ and let m denote Lebesgue measure on I. We consider piecewise expanding maps $T\colon I \to I$, satisfying the following assumptions:

(i) There is a number $\lambda > 1$, and a finite partition $\mathcal{P}$ of I into intervals, such that the restriction of T to any interval in $\mathcal{P}$ can be extended to a C^2-function on the closure, and $|T'| > \lambda$ on this interval.

(ii) T has an absolutely continuous invariant measure μ with respect to which T is mixing.

(iii) T has the property of being weakly covering, as defined by Liverani in [12], namely that there exists an n_0 such that for any element $\alpha \in \mathcal{P}$

$$\bigcup_{j=0}^{n_0} T^j(\alpha) = I.$$

For any $n \ge 0$ we define the partition $\mathcal{P}_n = \mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P}$. The partition elements of $\mathcal{P}_n$ are called n-cylinders, and $\mathcal{P}_n$ is called the partition of I into n-cylinders.

We prove the following two theorems.

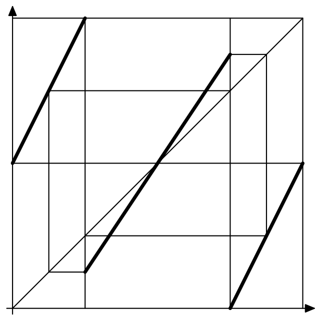

Figure 1: The graph of T.

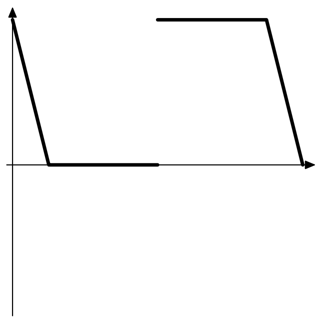

Figure 2: The graph of χ.

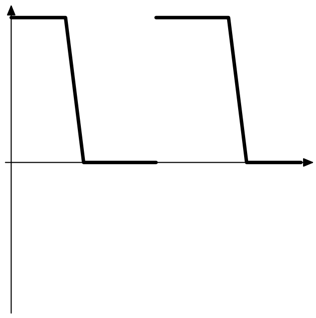

Figure 3: The graph of $\chi \circ T$.

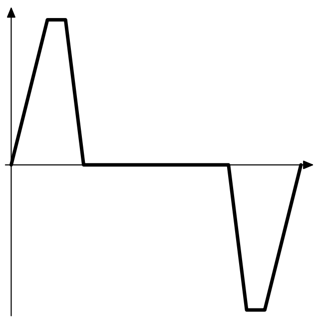

Figure 4: The graph of $\phi = \chi \circ T - \chi$.

Theorem 1. *Let (T, I, μ) be a piecewise expanding map satisfying assumptions (i), (ii) and (iii). Let $\phi\colon I \to \mathbb{R}$ be a Hölder continuous function, such that $\phi = \chi \circ T - \chi$ for some measurable function χ, with $e^{-\chi} \in L_1(m)$. Then there exists a function χ_0 such that χ_0 has bounded variation and $\chi_0 = \chi$ almost everywhere.*

For the next theorem we need some more definitions. Let A be a set, and denote by $\operatorname{int} A$ the interior of the set A. We assume that the open sets $T(\operatorname{int} \alpha)$, where α is an element in $\mathcal{P}$, cover $\operatorname{int} I$.

We will now define a new partition $\mathcal{Q}$. For a point x in the interior of some element of $\mathcal{P}$, we let $Q(x)$ be the largest open set such that for any $x_2 \in Q(x)$, and any m-cylinder C_m, there are points $(y_{1,k})_{k=1}^n$ and $(y_{2,k})_{k=1}^n$, such that $y_{1,k}$ and $y_{2,k}$ are in the same element of $\mathcal{P}$, $T(y_{i,k+1}) = y_{i,k}$, $T(y_{1,1}) = x$, $T(y_{2,1}) = x_2$, and $y_{1,n}, y_{2,n} \in C_m$. (This forces $n \geq m$.)

Note that if $Q(x) \cap Q(y) \neq \emptyset$, then for $z \in Q(x) \cap Q(y)$ we have $Q(z) = Q(x) \cup Q(y)$. We let $\mathcal{Q}$ be the coarsest collection of connected sets, such that any element of $\mathcal{Q}$ can be represented as a union of sets $Q(x)$.

Theorem 2. *Let (T, I, μ) be a piecewise expanding map satisfying assumptions (i), (ii) and (iii). If $\phi\colon I \to \mathbb{R}$ is a continuously differentiable function, such that $\phi = \chi \circ T - \chi$ for some function χ with $e^{-\chi} \in L_1(m)$, then there exists a function χ_0 such that χ_0 is continuously differentiable on each element of $\mathcal{Q}$ and $\chi_0 = \chi$ almost everywhere. If T' is constant on the elements of $\mathcal{P}$, then χ_0 is piecewise C^k on $\mathcal{Q}$ if ϕ is in C^k. If for each r, $\frac{1}{(T^r)'}$ is in C^k with derivatives up to order k uniformly bounded, then χ_0 is piecewise C^k on $\mathcal{Q}$ if ϕ is in C^k.*

It is not always clear how big the elements in the partition $\mathcal{Q}$ are. The following lemma gives a lower bound on the diameter of the elements in $\mathcal{Q}$.

Lemma 1. Assume that the sets $\{\, T(\operatorname{int} \alpha) : \alpha \in \mathcal{P} \,\}$ cover $(0, 1)$. Let δ be the Lebesgue number of the cover. Then the diameter of $\mathcal{Q}(x)$ is at least $\delta/2$ for all x.

Proof. Let C_m be a cylinder of generation m. We need to show that for some $n \geq m$ there are sequences $(y_{1,k})_{k=1}^n$ and $(y_{2,k})_{k=1}^n$ as in the definition of $\mathcal{Q}$ above.

Take n_0 such that $\mu(T^{n_0}(C_m)) = 1$. Write C_m as a finite union of cylinders of generation n_0, $C_m = \bigcup_i D_i$. Then $R := [0, 1] \setminus T^{n_0}(\cup_i \operatorname{int} D_i)$ consists of finitely many points. Let ε be the smallest distance between two of these points.

Let I_δ be an open interval of diameter δ. Let n_1 be such that $\delta \lambda^{-n_1} < \varepsilon$. Consider the full pre-images of I_δ under T^{n_1}. By the definition of δ, there is

at least one such pre-image, and any such pre-image is of diameter less than ε. Hence any pre-image contains at most one point from R.

If the pre-image does not contain any point of R, then I_δ is contained in some element of $\mathcal{Q}$ and we are done. Assume that there is a point z in I_δ corresponding to the point of R in the pre-image of I_δ. Assume that z is in the right half of I_δ. The case when z is in the left part is treated in a similar way. Take a new open interval J_δ of length δ, such that the left half of J_δ coincides with the right half of I_δ.

Arguing in the same way as for I_δ, we find that a pre-image of J_δ contains at most one point of R. If there is no such point, or the corresponding point $z_J \in J_\delta$ is not equal to z, then $I_\delta \cup J_\delta$ is contained in an element in $\mathcal{Q}$ and we are done.

It remains to consider the case $z = z_J$. Let $I_\delta = (a, b)$ and $J_\delta = (c, d)$. Then the intervals (a, z) and (z, d) are both of length at least $\delta/2$, and both are contained in some element of $\mathcal{Q}$. This finishes the proof. $\square$

Corollary 1. *If $\beta > 1$ and $T\colon x \mapsto \beta x \pmod 1$ is a β-transformation then clearly T is weakly covering and $\mathcal{Q} = \{(0, 1)\}$, so in this case Theorem 2 and Theorem 1 of [15] imply that χ_0 is in C^k if ϕ is in C^k.*

Remark 1. If $T\colon x \mapsto \beta x + \alpha \pmod 1$ is an affine β-transformation, then $\mathcal{Q} = \{(0, 1)\}$, and hence if $e^{-\chi}$ is in $L_1(m)$ then χ has a C^k version.

5 Proof of Theorem 1

We continue to assume that (T, I, μ) is a piecewise expanding map satisfying assumptions (i), (ii) and (iii). For a function $\psi\colon I \to \mathbb{R}$ we define the weighted transfer operator $\mathcal{L}_\psi$ by

$$\mathcal{L}_\psi f(x) = \sum_{T(y)=x} e^{\psi(y)} \frac{1}{|\mathrm{d}_y T|} f(y).$$

The proof is based on the following two facts, that can be found in Hofbauer and Keller's papers [8, 9]. The first fact is

> There is a function $h \geq 0$ of bounded variation such that if $f \in L^1$ with $f \geq 0$ and $f \neq 0$, then $\mathcal{L}_0^n f$ converges to $h \int f \,\mathrm{d}m$ in L^1. (2)

The second fact is

> Let $f \in L^1$ with $f \geq 0$ and $f \neq 0$ be fixed. There is a function $w \geq 0$ with bounded variation, a measure ν, and a number $a > 0$, depending on ϕ, such that
>
> $$a^n \mathcal{L}_\phi^n f \to w \int f \, \mathrm{d}\nu, \tag{3}$$
>
> in L^1.

For f of bounded variation, these facts are proved as follows. Theorem 1 of [8] gives us the desired spectral decomposition for the transfer operator acting of functions of bounded variation. Proposition 3.6 of Baladi's book [3] gives us that there is a unique maximal eigenvalue. This proves the two facts for f of bounded variation. The case of a general f in L^1 follows since such an f can be approximated by functions of bounded variation.

Using that T is weakly covering, we can conclude by Lemma 4.2 in [12], that $h > \gamma > 0$. The proof of this fact in [12] goes through also for w, and so we may also conclude that $w > \gamma > 0$.

Let us now see how Theorem 1 follows from these facts. The following argument is analogous to the argument used by Pollicott and Yuri in [23] for β-expansions. We first observe that $\phi = \chi \circ f - \chi$ implies that

$$\begin{aligned} \mathcal{L}_\phi^n 1(x) &= \sum_{T^n(y)=x} e^{S_n \phi(y)} \frac{1}{|\mathrm{d}_y T^n|} = \sum_{T^n(y)=x} e^{\chi(T^n y) - \chi(y)} \frac{1}{|\mathrm{d}_y T^n|} \\ &= e^{\chi(x)} \sum_{T^n(y)=x} e^{-\chi(y)} \frac{1}{|\mathrm{d}_y T^n|} = e^{\chi(x)} \mathcal{L}_0^n e^{-\chi}(x). \end{aligned}$$

Since $a^n \mathcal{L}_\phi^n 1 \to w$ and $e^{-\chi} \mathcal{L}_\phi^n 1 = \mathcal{L}_0^n e^{-\chi} \to h \int e^{-\chi} \, \mathrm{d}m$ we have that $a^n \mathcal{L}_\phi^n 1$ converges to w in L^1 and $\mathcal{L}_\phi^n 1$ converges to $h e^\chi \int e^{-\chi} \, \mathrm{d}m$ in L^1. By taking a subsequence, we can achieve that the convergences are a.e. Therefore, we must have $a = 1$ and

$$w(x) = e^{\chi(x)} h(x) \int e^{-\chi} \, \mathrm{d}m, \quad \text{a.e.}$$

It follows that

$$\chi(x) = \log w(x) - \log \int e^{-\chi} \, \mathrm{d}m - \log h(x),$$

almost everywhere. Since h and w are bounded away from zero, their logarithms are of bounded variation. This proves the theorem.

6 Proof of Theorem 2

We first note that it is sufficient to prove that χ_0 is continuously differentiable on elements of the form $Q(x)$.

Let x and y satisfy $T(y) = x$. Then by $\phi = \chi \circ T - \chi$ we have $\chi(x) = \phi(y) + \chi(y)$.

Let x_1 be a point in an element of $\mathcal{Q}$, and take $x_2 \in Q(x_1)$. We choose pre-images $y_{1,j}$ and $y_{2,j}$ of x_1 and x_2 such that $T(y_{i,1}) = x_i$ and $T(y_{i,j}) = y_{i,j-1}$. We then have

$$\chi(x_1) - \chi(x_2) = \sum_{j=1}^{n} \big(\phi(y_{1,j}) - \phi(y_{2,j})\big) + \chi(y_{1,n}) - \chi(y_{2,n}).$$

We would like to let $n \to \infty$ and conclude that $\chi(y_{1,n}) - \chi(y_{2,n}) \to 0$. By Theorem 1 we know that χ has bounded variation. Assume for contradiction that no matter how we choose $y_{1,j}$ and $y_{2,j}$ we cannot make $|\chi(y_{1,n}) - \chi(y_{2,n})|$ smaller than some $\varepsilon > 0$. Let m be large and consider the cylinders of generation m. For any such cylinder C_m, we can choose $y_{1,j}$ and $y_{2,j}$ such that $y_{1,n}$ and $y_{2,n}$ both are in C_m. Since $|\chi(y_{1,n}) - \chi(y_{2,n})| \geq \varepsilon$, the variation of χ on C_m is at least ε. Summing over all cylinders of generation m, we conclude that the variation of χ on I is at least $N(m)\varepsilon$. Since m is arbitrary and $N(m) \to \infty$ as $m \to \infty$, we get a contradiction to the fact that χ is of bounded variation.

Hence we can make $|\chi(y_{1,n}) - \chi(y_{2,n})|$ smaller that any $\varepsilon > 0$ by choosing $y_{1,j}$ and $y_{2,j}$ in an appropriate way. We conclude that

$$\chi(x_1) - \chi(x_2) = \sum_{j=1}^{\infty} \big(\phi(y_{1,j}) - \phi(y_{2,j})\big).$$

If $x_1 \neq x_2$ then $y_{1,j} \neq y_{2,j}$ for all j, and we have

$$\frac{\chi(x_1) - \chi(x_2)}{x_1 - x_2} = \sum_{j=1}^{\infty} \frac{\phi(y_{1,j}) - \phi(y_{2,j})}{y_{1,j} - y_{2,j}} \frac{y_{1,j} - y_{2,j}}{x_1 - x_2}.$$

Clearly, the limit of the right hand side exists as $x_2 \to x_1$, and is

$$\sum_{j=1}^{\infty} \phi'(y_{1,j}) \frac{1}{(T^j)'(y_{1,j})}.$$

The series converges since $|(T^j)'| > \lambda^j$. This shows that $\chi'(x_1)$ exists and satisfies

$$\chi'(x_1) = \sum_{j=1}^{\infty} \phi'(y_{1,j}) \frac{1}{(T^j)'(y_{1,j})}. \tag{4}$$

If T' is constant on the elements of $\mathcal{P}$, then (4) implies that χ is in C^k provided that ϕ is in C^k.

Let us now assume that $\frac{1}{(T^r)'}$ is in C^k with derivatives up to order k uniformly bounded in r. We proceed by induction. Let $g_n = \frac{1}{(T^n)'}$. Assume that

$$\chi^{(m)}(x) = \sum_{n=1}^{\infty} \psi_{n,m}(y_n) g_n(y_n), \tag{5}$$

where $(\psi_{n,m})_{n=1}^{\infty}$ is in C^{n-m} with derivatives up to order $n-m$ uniformly bounded. Then

$$\chi^{(m+1)}(x) = \sum_{n=1}^{\infty} \big(\psi'_{n,m}(y_n) g_n(y_n) + \psi_{n,m}(y_n) g'_n(y_n)\big) g_n(y_n) = \sum_{n=1}^{\infty} \psi_{n,m+1} g_n(y_n).$$

This proves that there are uniformly bounded functions $\psi_{n,m}$ such that (5) holds for $1 \leq m \leq k$. The series in (5) converges uniformly since g_n decays with exponential speed. This proves that χ is in C^k. $\square$

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